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A fourth-order topological invariant of magnetic or vortex lines

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Abstract

A fourth-order topological invariant for non-dissipative magnetic or vortex configurations with zero helicity is constructed. It is an integral form of the two-link Sato–Levine topological invariant. Geometrically, the invariant is determined by the self-linking number of the curve of intersection of Seifert surfaces pulled on two linked flux tubes.

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1. Introduction

One of the classical problems of geometry and topology: understanding the difference between knots and links, has found some interesting applications in the physics of polymers and quantum field theory. Now finding "equivalent" vortex or magnetic lines is a focus of interest in classical hydrodynamics and electrodynamics. The use of the helicity invariant, which measures the total linking of pairs of vortex or magnetic lines, has been very successful. For example, this invariant is an essential component of dynamo theories explaining the origin of the planetary, stellar and galactic magnetic fields [Moffatt 1978, Parker 1979, Krause and Rädler 1980, Zeldovich et al. 1983], and it constrains the energy of relaxing magnetized plasma configurations [Taylor 1974, Freedman 1988, Moffatt 1990].

There are however many topologically more complicated configurations, in particular those having zero helicity. In knot theory they are distinguished by different invariants. To formulate these invariants the power of group theory is normally used. This

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approach generates some polynomials, the coefficients of which are the invariants. The Alexander polynomials were known early, then the Jones polynomials, and recently the more general Vassiliev invariants and corresponding polynomials were discovered [Vassiliev 1990, Arnold 1992]. It is still not clear how those polynomials can be used in physics, especially in hydrodynamics or magnetohydrodynamics. The problem is to express the topological invariants in terms of the observed parameters. At the moment this problem has been solved effectively only for the helicity, which can be represented as an integral with an integrand that is of second-order in the field amplitude. An integral form for a third-order invariant (the Borromean rings) has been suggested by Berger [Berger 1990] and extended to higher-order odd invariants by Evans and Berger [Evans and Berger 1992].

In this paper an attempt is made to construct "a physical form" for a topological invariant for a two flux tube configuration with zero helicity. A well-known example of such a configuration is the Whitehead link. The corresponding topological invariant in knot theory was found by Levine and Sato [Sato 1984]. The main result of this paper is the presentation of the Sato–Levine invariant in integral form with the integrand expressed through vector-potentials of the magnetic field in flux tubes. The integrand is of fourth order in the field amplitude so one can refer to the integral as "fourth-order topological invariant".

2. Framing and Seifert's surfaces

Consider closed field lines concentrated into thin flux tubes. The flux tube is assumed oriented, i.e. a direction of the vorticity or magnetic field along the tube is specified. More complicated morphologies of the magnetic fields can be studied by separating the space into regions bounded by magnetic surfaces, $\boldsymbol{B} \cdot \boldsymbol{n}|_s = 0$ [Berger and Field 1984]. The analysis of ergodic lines can be reduced to the case of closed tubes [Arnold 1974, Arnold and Khesin 1992].

It is natural to use a framing, i.e. a set of normals to the axis of the flux tube, and natural coordinate lines on the flux tube, i.e. meridians and longitudes. A meridian is a closed curve which bounds a cross-sectional disk of the flux tube. The meridian is a uniquely defined coordinate line because any two meridians are isotopically equivalent, i.e. can be made to coincide by deformation (see, for example, [Rolfson 1976]). However the definition of longitude involves a choice. There are an infinite number of isotopically different longitudes, depending on how many times a longitude curve twists around the flux tube before it returns to the starting point. It is always possible to choose the longitude in such a way that its twisting will compensate the writhe (kinks) of the flux tube. In other words, among all framings there is a preferred one which has self-linking number of zero (see, for example, [Sato 1984]).

Consider now the preferred longitude on the flux tube and join to it, along the normals, a surface having the flux tube as its boundary. This surface, called in topology the Seifert surface (see, for example, [Milnor 1966] and [Rolfson 1976]), is compact, connected and orientable (has two sides). In contrast to any other surfaces, the Seifert surface has a remarkable property: the self-linking number (helicity) of its boundary is zero. In spite of this constraint there still are, evidently, an infinite number of Seifert surfaces. In physical terms, the Seifert surface for a flux tube or a set of flux tubes can be constructed with the help of level-surfaces of the vector-potential created outside the flux tubes [Akhmetiev and Ruzmaikin 1992]. The vector-potential in general is multivalued; however, there are some regions in which the surfaces are defined by single-valued functions. This fact will be used in our construction below.

3. A topological invariant of two-component links

With the help of Seifert surfaces it is possible to construct a topological invariant for configurations consisting of two flux tubes linked in such a way that the helicity of the configuration is zero [Sato 1984]. An example of such a link is the Whitehead link [Crowell and Fox 1963], see Fig. 1.

Consider a link consisting of a pair, say U_i and U_j, of mutually disjoint, closed magnetic or vortex flux tubes. Each flux tube bounds a Seifert surface. Links for which this pair of surfaces do not intersect are trivial (equivalent to two separated flux tubes). In the general case, each Seifert surface bounding a flux tube intersects the other flux tube. However when the linking number L_{ij} of these two flux tubes is zero, one can find Seifert surfaces which do not intersect flux tubes. The intersection of these two Seifert surfaces has the form of a closed framed curve. The self-linking number of this framed curve is a topological invariant [Milnor 1966]. The invariant is independent of the particular pair of the Seifert surfaces, as was pointed out by Levine and Sato [Sato 1984].

To demonstrate this construction consider two flux tubes linked as in the Whitehead link. Pull a Seifert surface in the form of a disk on the "large" flux tube with "twist over-pass" near the clasp (see [Cochran 1990], and Fig. 2a). The Seifert surface pulled

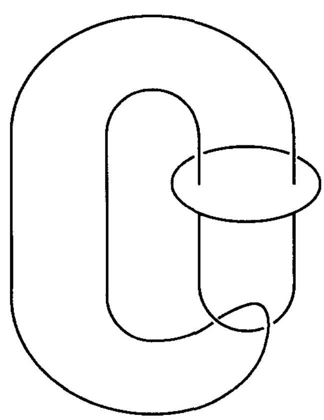

Fig. 1. The Whitehead link as an example of a non-trivial link with zero helicity.

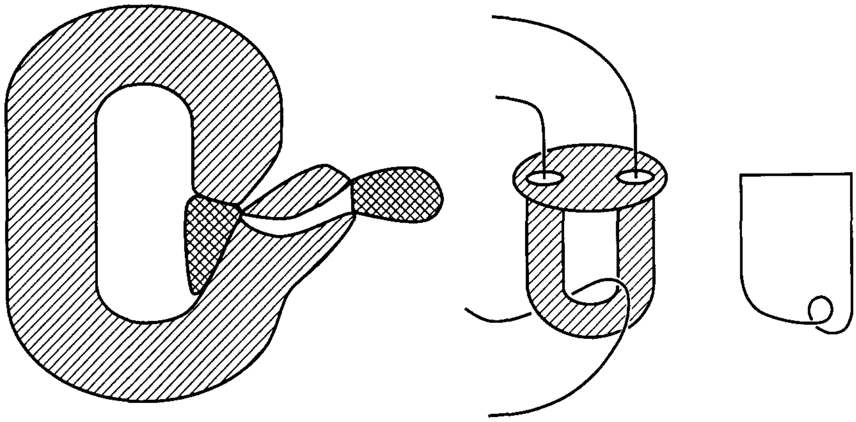

Fig. 2. The Seifert surfaces of the two flux tubes in the Whitehead link (presented separately in (a) and (b)) intersect along the self-linked curve shown on the right side of the picture. Near the clasp the "twist over-pass" surface is used [Cochran 1990]

on the other flux tube has the form of a handle to avoid intersecting or touching the other flux tube (Fig. 2b).

It is easy to see that this surface will intersect the first Seifert surface along a closed twisted curve having non-zero self-linking number. This number is identified with the invariant under consideration, which will be denoted as W.

Thus, in the construction of this high-order topological invariant, the concept of the linking number is used again. In this case, however, it is applied not directly to the flux tubes, but to a closed line of intersection of two Seifert surfaces pulled on the flux tubes.

4. Integral form of the two-link invariant

Let us construct this invariant through vector-potentials created by the field concentrated in two unlinked flux tubes. Let $\boldsymbol{A}_i = \nabla\phi_i$ and $\boldsymbol{A}_j = \nabla\phi_j$ be the vector potentials outside the flux tubes. In so far as the flux tubes U_i and U_j are unlinked, the functions ϕ_i and ϕ_j are single-valued inside U_j and U_i, respectively. We will assume that all functions under consideration are smooth. The Seifert surface over the first flux tube is ϕ_i=const., and it is ϕ_j=const. for the second flux tube. The vector-potentials $\boldsymbol{A}_i$ and $\boldsymbol{A}_j$ are normal to the corresponding Seifert surfaces. The vector-potential created by the flux tube U_i inside the flux tube U_j is a gradient of some scalar function, $\boldsymbol{A}_i = \nabla\phi_i$. And the vector-potential created by the flux tube U_j inside the flux tube U_i is $\boldsymbol{A}_j = \nabla\phi_j$. Consider the vector

$$\boldsymbol{G} = \boldsymbol{A}_i \times \boldsymbol{A}_j - \phi_j \boldsymbol{B}_i + \phi_i \boldsymbol{B}_j. \quad (1)$$

The vector $\boldsymbol{G}$ is divergence-free outside the flux tubes since

$$\nabla \cdot (A_i \times A_j) = \nabla \cdot (\nabla\phi_i \times \nabla\phi_j) = (\nabla \times \nabla\phi_i) \cdot \nabla\phi_j - \nabla\phi_i \cdot (\nabla \times \nabla\phi_j) = 0.$$

It is easy to check that $\nabla \cdot G$ also vanishes inside the flux tubes; that is why the two last terms in (1) are needed. According to the Poincaré lemma there exists a vector potential F such that $G = \nabla \times F$. The invariant under consideration, which may be interpreted as a self-linking number of the vector field G, can now be defined as follows:

$$W(i,j) = \int (G \cdot F - \phi_j^2 A_i \cdot B_i - \phi_i^2 A_j \cdot B_j)\, d^3x. \tag{2}$$

The integral is taken over all space while the last two terms are non-zero only inside the flux tubes.

The quantity $W(i,j)$ is gauge-invariant. In fact, let $A_i \to A_i + \nabla f$, where f is an arbitrary single-valued function. Then $G \to G + \nabla f \times A_j + fB_j$. The corresponding change in the vector-potential F is fA_j. By use of these expressions and the Gauss theorem one can directly check that the integral (2) does not change under this gauge transformation:

$$\begin{aligned} W \to W &+ \int (Gf \cdot A_j + F \cdot \nabla \times (fA_j) + f^2 A_j \cdot B_j + \phi_j^2 B_j \cdot \nabla f \\ &- 2f\phi_i A_j \cdot B_j - f^2 A_j \cdot B_j) d^3x \\ = W &+ \int (2Gf \cdot A_j + \phi_j^2 B_j \cdot \nabla f - 2f\phi_i A_j \cdot B_j) d^3x \\ = W &+ \int B_i \cdot \nabla(f\phi_j^2) d^3x = W. \end{aligned}$$

Similarly one can check that the integral (2) is invariant under the change $A_j \to A_j + \nabla g$.

Let us prove now that this integral representation is equivalent to the Sato–Levine geometrical construction.

5. Proof of equivalence

From a simple geometrical point of view the equivalence between the integral (2) and the Sato–Levine invariant can be explained as follows. Outside the flux tubes only the first term in (1) can be non-zero. This means that the vector G there is perpendicular to both vector-potentials A_i and A_j, i.e. it defines a closed curve of intersection between two Seifert surfaces having as their normals A_i and A_j. The frame along this curve of intersection is determined by the normal to one of these surfaces, i.e. by A_i or A_j. Hence the helicity of the vector G is the self-linking number of the curve of intersection, i.e. the Sato–Levine invariant.

The topological proof is more sophisticated. Below we sketch it. Let us assume that the potentials A_i and A_j vanish at infinity. Then we can work in the compactification of the three-dimensional space S^3 obtained by joining the infinite point to R^3. Let K_0^3 be the complement space to the two-link. It is obtained from S^3 by subtracting the volume occupied by the two flux tubes. The idea of the proof is to show that a mapping of K_0^3 (modified by a surgery) into a two-dimensional torus $T^2 = T_i \times T_j$, where T_i and T_j are the toroidal surfaces of the flux tubes, has a Hopf invariant (2) equal to the self-linking number of the curve of intersection.

The boundary of K_0^3 consists of the two toroidal surfaces of the flux tubes. Fix on each surface a coordinate system, the meridian μ and the preferred longitude λ, i.e. the longitude which has zero self-linking number. Let us make for each torus the following reconstruction (the so called Dehn surgery, see for example [Browder 1972]). Cut the torus along a longitude and a meridian, then paste one of the λ curves into one of the μ curves, and the other μ curve into the remaining λ curve but in the opposite direction. (One may treat this construction as an abstract transformation, or visualize it by looking from four-dimensional space. Formally, this is Dehn's surgery with a 2-d matrix having zeros on the diagonal and 1 and -1 for off-diagonal terms.) The modified complement space with the repasted boundary is called K^3.

Consider now a mapping of K^3 into T^2 in the form $F = \phi_i \times \phi_j$ outside the flux tubes and F is arbitrary inside the flux tubes. Let L be a curve formed by the inverse image of a point on T^2. This curve is framed by the inverse image of a vector on T^2. We can consider L as the intersection of the Seifert surfaces with the frame along L being a tangent frame along one of the surfaces.

The Hopf invariant of the mapping f is determined by the formula $\int_{K^3} s \wedge^* \omega$ [Dubrovin et al. 1979], where s is the 1-form defined by $ds = {}^*\omega$, and ω is a 2-form on T^2 defined in such away that its inverse image is ${}^*\omega = d\phi_i \wedge d\phi_j = A_i \wedge A_j$. It is noted in the textbook by Dubrovin et al. that the invariant is independent of the choice of s, and gauge invariant, i.e. invariant under the transformation ω to $\omega + d\chi$ where χ is an arbitrary 1-form. Formula (2) for W is a direct extension of the Hopf invariant. The extra terms in (2), compared to the usual helicity, appear because in our case the mapping includes a non-compact manifold (the external domain of the flux tubes).

6. Discussion

The representation (2) of the two-link topological invariant in terms of quantities familiar to physicists is a step towards making the high-order topological invariants more useful in applications. However some important questions have to be answered before we can try to use this invariant. One of them is: "Is this topological invariant a Hopf invariant, i.e. a conservation law, of ideal MHD?" The answer to this question is apparently "yes" because the Sato-Levine invariant, and consequently W, is invariant under deformations of space.

Another question concerns the role of diffusivity. In real hydrodynamics and magnetohydrodynamics with viscosity or diffusion there is, in general, no concept of vortex or magnetic lines. However, when the diffusivities are small, or more correctly, the dimensionless Reynolds numbers are large, it is still reasonable to consider the field lines, taking into account diffusion effects only in regions of strong field gradients. In these regions magnetic or vortex lines can reconnect. It is known that reconnections almost conserve the helicity integral in the limit of small diffusivity [Taylor 1974]. Transformations of high-order topological invariants under reconnections are discussed in [Akhmetiev and Ruzmaikin 1992, Ruzmaikin and Akhmetiev 1994].

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